Math 132: Differential Topology

§ Derivatives and tangents

- For any smooth map $f: U \to V$ where $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$ are open,

  the derivative at $x \in U$ is the linear map

  $$df_x : \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$$

  defined by the directional derivative

  $$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}, \quad h \in \mathbb{R}^k.$$

  It can be represented by the $\ell \times k$ Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{i=1,\cdots,\ell, \\ j=1,\cdots,k}}$.

  (After translation, this is the best approximation of f by a (nonhomogeneous) linear map, at x.)

- By chain rule, for any commutative triangle of smooth maps

  $$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V \arrow[r, "g"] & W \end{tikzcd},$$

  we have a commutative triangle of linear maps

  $$\begin{tikzcd} \mathbb{R}^k \arrow[r, "df_x"] \arrow[rr, bend right, "d(g \circ f)_x"'] & \mathbb{R}^\ell \arrow[r, "dg_{f(x)}"] & \mathbb{R}^m \end{tikzcd}.$$

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(Dimension is a diffeomorphism invariant)

Thm If f is a diffeomorphism between open sets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$, then k must be equal to ℓ, and $df_x : \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$ must be nonsingular.

proof) Since $f^{-1} \circ f = id_U$, $d(f^{-1})_{f(x)} \circ df_x = id_{\mathbb{R}^k}$.

Similarly $df_x \circ d(f^{-1})_{f(x)} = id_{\mathbb{R}^\ell}$. Thus df_x has a two-sided inverse, and it follows that $k = \ell$. ■

Def For a smooth m-manifold $M \subset \mathbb{R}^k$, choose a parametrization

$$g : U \longrightarrow M \subset \mathbb{R}^k$$ of a neighborhood $g(U)$ of x in M,

with $g(u) = x$.

Set the tangent space of M at x to be

$$T_x M := dg_u(\mathbb{R}^m),$$ i.e. the image of $dg_u : \mathbb{R}^m \longrightarrow \mathbb{R}^k$.

Prop This definition does not depend on the choice of parametrization.

proof) If $h : V \longrightarrow M \subset \mathbb{R}^k$ is another parametrization of a neighborhood $h(V)$ of x in M, with $h(v) = x$, then

$$\begin{tikzcd} & \mathbb{R}^k & \\ U' \arrow[ur, "g"] \arrow[rr, "\cong", "h^{-1}\circ g"'] & & V' \arrow[ul, "h"'] \end{tikzcd}$$

where $U' = g^{-1}(g(U) \cap h(V))$,
$V' = h^{-1}(g(U) \cap h(V))$

so

$$\begin{tikzcd} & \mathbb{R}^k & \\ \mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "\cong", "d(h^{-1}\circ g)_u"'] & & \mathbb{R}^m \arrow[ul, "dh_v"'] \end{tikzcd}$$. ■

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Prop T_xM is m-dimensional (i.e. $dg_u : \mathbb{R}^m \to \mathbb{R}^k$ is injective).

proof) Since $g^{-1} : g(U) \to U$ is smooth, there is an open set W containing x and a smooth map $F : W \to \mathbb{R}^m$ that coincides with g^{-1} on $W \cap g(U)$. Then we have

$$\begin{tikzcd}
 & W \arrow[dr, "F"] & \\
U' \arrow[ur, "g"] \arrow[rr, hook] & & \mathbb{R}^m
\end{tikzcd}$$

where $U' := g^{-1}(W \cap g(U))$,

so

$$\begin{tikzcd}
 & \mathbb{R}^k \arrow[dr, "dF_x"] & \\
\mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "\cancel{\mathrm{id}}"] & & \mathbb{R}^m
\end{tikzcd}$$

. $\blacksquare$

Def For a smooth map $f : M \to N$ between smooth manifolds $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$, the derivative

$$df_x : T_xM \to T_{f(x)}N$$

is defined by $df_x(v) = dF_x(v)$, $v \in T_xM$, for any smooth map $F : W \to \mathbb{R}^\ell$ that coincides with f on $W \cap M$.

Prop $dF_x(v) \in T_{f(x)}N$ and it does not depend on the particular choice of F.

proof) Choose parametrizations

$$g : U \to M \subset \mathbb{R}^k, \quad h : V \to N \subset \mathbb{R}^\ell$$

for neighborhoods $g(U)$ of x in M and $h(V)$ of $f(x)$ in N.

Replacing U by smaller set if necessary, we may assume that

$$g(U) \subset W \text{ and that } f \circ g(U) \subset h(V).$$

Then we can consider the commutative diagram

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbb{R}^\ell \\ U \arrow[u, "g"] \arrow[r, "h^{-1}\circ f\circ g"] & V \arrow[u, "h"'] \end{tikzcd}$$

of smooth maps between open sets.

Taking derivatives,

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^\ell \\ \mathbb{R}^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1}\circ f\circ g)_u"] & \mathbb{R}^n \arrow[u, "dh_v"'] \end{tikzcd},$$

and it is clear that $dF_x(T_xM) \subset T_{f(x)}N$.

Moreover, $df_x = dh_v \circ d(h^{-1}\circ f\circ g)_u \circ (dg_u)^{-1}$, so it is independent of F. Therefore, $df_x : T_xM \to T_{f(x)}N$ is well-defined. ■

<u>Rmk</u> As before, chain rule still holds

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & N \arrow[r, "g"] & P \end{tikzcd} \rightsquigarrow \begin{tikzcd} T_xM \arrow[r, "df_x"] \arrow[rr, bend right, "d(g\circ f)_x"'] & T_yN \arrow[r, "dg_y"] & T_zP \end{tikzcd},$$

and for any diffeo $f : M \to N$, $df_x : T_xM \to T_yN$ is an iso.